

On the integration of Poisson homogeneous spaces

Francesco Bonechi^a, Nicola Ciccoli^b, Nicola Staffolani^c and
Marco Tarlini^a

^a*I.N.F.N. Sezione di Firenze,*

Via G. Sansone 1, 50019 Sesto Fiorentino - Firenze, Italy

email: <name> @fi.infn.it

^b*Dipartimento di Matematica e Informatica,*

Via Vanvitelli 1, 06123 - Perugia, Italy

email: ciccoli@dipmat.unipg.it

^c*Dipartimento di Fisica,*

Via G. Sansone 1, 50019 Sesto Fiorentino - Firenze, Italy

email: staffolani@fi.infn.it

Abstract

We study a reduction procedure for describing the symplectic groupoid of a Poisson homogeneous space obtained by quotient of a coisotropic subgroup. We perform it as a reduction of the Lu-Weinstein symplectic groupoid integrating Poisson Lie groups, that is suitable even for the non-complete case.

Keywords: Poisson geometry, Symplectic groupoids, Poisson homogeneous spaces, Poisson-Lie groups, Coisotropic subgroups, Geometric quantization.

MSC: 53D05, 17B63, 22A22, 53D17, 53D50

JGP subject code: Symplectic geometry

1 Introduction

Symplectic groupoids were introduced by Karasev and Weinstein in the 80's, [9, 22] as a tool to quantize Poisson manifolds. They immediately became objects of independent math interests and one of the cornerstones of Poisson geometry. Our knowledge on their role dramatically improved after the work by Cattaneo-Felder [2], interpreting them as the phase space of the Poisson sigma model, and Crainic-Fernandes [5] on the integrability of algebroids. On the contrary, quantum aspects were much less studied: in [29] P.Xu and A.Weinstein defined the right notion of prequantization. Such a prequantization can be explicitly constructed by reducing the prequantization of the phase space of the Poisson sigma model, as shown in [1]. In [24] the so-called noncommutative torus was recovered by the geometric quantization of the symplectic groupoid integrating the underlying symplectic structure. Very recently, a notion of polarization of symplectic groupoids has been introduced in [8]. Very roughly, the idea is to associate to any Poisson manifold a C^* -algebra constructed out of the space of polarized sections. This idea realizes a fundamental pattern in noncommutative geometry, where according to the Gelfand-Naimark theorem the noncommutative counterpart of the algebra of continuous functions on a (compact) manifold is a C^* -algebra.

During the same years, following the impulse of A. Connes, noncommutative geometry evolved trying to give an axiomatic description of what a noncommutative manifold should be. The most studied examples can be collected in two main classes: *i*) the so-called θ -manifolds, where the underlying Poisson structure is determined by the action of an abelian group, like the celebrated noncommutative torus, and the Connes-Landi 4-sphere, see [3] and also [4]; *ii*) quantum groups and the associated homogeneous spaces. The θ -manifolds are well studied from many different points of view: the symplectic groupoid was described first in [27], their deformation quantization in [19] and the polarization of the symplectic groupoid in [8]. On the contrary, for spaces related to quantum groups less is known. The Poisson Lie groups, that are the semiclassical limit of quantum groups, were shown to be integrable in [14]. The problem of Poisson reduction of the symplectic groupoid was discussed in [26]: from these results the integration of Poisson homogeneous spaces that are quotient by Poisson Lie subgroups can be obtained. With a completely different approach, Poisson symmetric spaces (a definition which does not contain all covariant Poisson structures on symmetric spaces) were shown to be integrable in [28]. Nothing is known concerning the quantization of the symplectic groupoid.

Much information about quantum aspects comes from quantum group theory, especially for what concerns the study of C^* -algebras of basic examples of quantum groups and homogeneous spaces. It is an interesting program to look at these constructions from

the point of view of the quantization of the symplectic groupoid like the one proposed in [8].

When we started this project, we realized that there was no construction of symplectic groupoids integrating the most important examples of Poisson homogeneous spaces coming from the semiclassical limit of homogeneous spaces of quantum groups, like Podles spheres, odd spheres, quantum grassmanians. The most relevant properties of the underlying Poisson structures is that they can be obtained as quotients by coisotropic subgroups. The present paper is devoted to this construction and must be thought of as preliminary to the problem of quantization, that we hope to address in the future. While we were finishing this paper, we were aware of [13], where a Poisson groupoid on any Poisson homogeneous space is presented. Moreover, conditions for the Poisson structure to be nondegenerate, so giving a symplectic groupoid, are discussed. The paper [13] covers a large part of our results, that we obtained independently; in particular in the complete case it gives the symplectic groupoid that we describe in Theorem 12. Nevertheless, since the two approaches are different we think that our paper can help to clarify some issues. The differences between our paper and [13] come out in the discussion of the noncomplete case (*i.e.* when dressing vector fields are not complete). In fact under a weaker hypothesis about the integrability of dressing vector fields (that we call *relative completeness*), we always obtain a symplectic groupoid. In concrete examples we show that the constructions are different.

This is the plan of the paper. In section 2 we recall very basic facts about Poisson manifolds and symplectic groupoids, mainly to fix notations. In section 3 we recall basic facts of Poisson Lie groups, following [12]. In section 4 we discuss the reduction procedure in terms of a moment map. When the Poisson Lie group is complete, the reduction is a straightforward analogue of the trivial case, where the action given by left multiplication of the subgroup can be lifted to a hamiltonian action on T^*G and the groupoid is obtained by Marsden-Weinstein reduction. Indeed if the Poisson Lie group G is complete then the left multiplication can be lifted to the groupoid and this action is hamiltonian in terms of Lu's momentum map. In the noncomplete case, the action can be lifted only as an infinitesimal action, and it is better to formulate it in terms of the symplectic action of the groupoid integrating the dual Poisson Lie group G^* . Nevertheless, if the subgroup satisfies the condition of relative completeness the procedure still works.

2 Preliminaries on Poisson manifolds

In this section we introduce the main definitions concerning the theory of Poisson manifolds and symplectic groupoids.

A *Poisson manifold* P is a smooth manifold provided with a bivector $\pi_P \in \Gamma(\Lambda^2 TP)$ satisfying $[\pi_P, \pi_P]_S = 0$, where $[\cdot, \cdot]_S$ is the Schouten bracket between multivector fields. The cartesian product $P = M \times N$ of two Poisson manifolds is a Poisson manifold with Poisson tensor $\pi_{M \times N} = \pi_M + \pi_N$. The Poisson bivector π_P defines a bundle map $\pi_P^\sharp : T^*P \rightarrow TP$ as $\langle \pi_P^\sharp(\omega_p), \nu_p \rangle = \langle \pi_P(p), \nu_p \wedge \omega_p \rangle$, for $p \in P$, $\omega_p, \nu_p \in T_p^*P$. A submanifold C of P is a *coisotropic submanifold* if $\pi_P^\sharp(N^*C) \subset TC$, where N^*C is the conormal bundle of C , $N_x^*C = \{\omega \in T_x^*P : \langle \omega, V \rangle = 0, \forall V \in T_x C\}$, for all $x \in C$. The generalized distribution defined by $\pi_P^\sharp(N^*C)$ is integrable and the space \underline{C} of coisotropic leaves, if smooth, is a Poisson manifold. A submanifold C is a *Poisson submanifold* if $\pi_P(c) \in \Lambda^2 T_c C$; it is coisotropic and the coisotropic foliation is trivial. A smooth map $\Psi : M \rightarrow N$ between two Poisson manifolds is a *Poisson map* if the Poisson tensors are Ψ -related. In [23] it is proven that if $\Psi : M \rightarrow N$ is a Poisson map and $\mathcal{O}_N \subset N$ is a symplectic leaf then $\Psi^{-1}(\mathcal{O}_N) \subset M$, whenever is a submanifold, is coisotropic.

Let $\mathcal{G} = (\mathcal{G}, \mathcal{G}_0, \alpha_{\mathcal{G}}, \beta_{\mathcal{G}}, m_{\mathcal{G}}, \iota_{\mathcal{G}}, \epsilon_{\mathcal{G}})$ be a Lie groupoid over the space of unities \mathcal{G}_0 , where $\alpha_{\mathcal{G}}, \beta_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{G}_0$ are the source and target maps, respectively, $m_{\mathcal{G}} : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ is the multiplication, $\iota_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{G}^{-1}$ is the inversion and $\epsilon_{\mathcal{G}} : \mathcal{G}_0 \rightarrow \mathcal{G}$ is the embedding of unities. Our conventions are that $(x_1, x_2) \in \mathcal{G}^{(2)}$ if $\beta_{\mathcal{G}}(x_1) = \alpha_{\mathcal{G}}(x_2)$. We say that \mathcal{G} is source simply connected (ssc) if $\alpha_{\mathcal{G}}^{-1}(m)$ is connected and simply connected for any $m \in \mathcal{G}_0$.

A *symplectic groupoid* is a Lie groupoid, which is equipped with a symplectic form, such that the graph of the multiplication is a lagrangian submanifold of $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$, where $\bar{\mathcal{G}}$ means \mathcal{G} with the opposite symplectic structure. There exists a unique Poisson structure on \mathcal{G}_0 such that $\alpha_{\mathcal{G}}$ and $\beta_{\mathcal{G}}$ are Poisson and anti-Poisson mappings, respectively. A Poisson manifold is said to be integrable if it is the space of units of a symplectic groupoid.

An equivalent characterization for a Lie groupoid \mathcal{G} to be a symplectic groupoid is that the symplectic form ω of \mathcal{G} be multiplicative, *i.e.* let $\text{pr}_{\mathcal{G}_1}, \text{pr}_{\mathcal{G}_2} : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ be respectively the projections onto the first and second factor, then $m_{\mathcal{G}}^* \omega = \text{pr}_{\mathcal{G}_1}^* \omega + \text{pr}_{\mathcal{G}_2}^* \omega$.

Following [18], we define the left action of \mathcal{G} on a manifold P with anchor $J : P \rightarrow \mathcal{G}_0$ a mapping from $\mathcal{G}_{\beta_{\mathcal{G}}} \times_J P = \{(x, p) \in \mathcal{G} \times P \mid \beta_{\mathcal{G}}(x) = J(p)\}$ to P , given by $(x, p) \rightarrow xp$ such that *i)* $J(xp) = \alpha_{\mathcal{G}}(x)$, *ii)* $(xy)p = x(yp)$, *iii)* $\epsilon(J(p))p = p$. In the case of P symplectic, the action of \mathcal{G} is called *symplectic* if the graph of the action $\{(x, p, xp), \beta_{\mathcal{G}}(x) = J(p)\}$ is lagrangian in $\mathcal{G} \times P \times \bar{P}$. In [18] it is shown that $J : P \rightarrow \mathcal{G}_0$ is a Poisson map. Symplectic reduction is defined as follows. The isotropy group $\mathcal{G}_m^m = \alpha_{\mathcal{G}}^{-1}(m) \cap \beta_{\mathcal{G}}^{-1}(m)$ of $m \in \mathcal{G}_0$ leaves invariant $J^{-1}(m)$ and $\mathcal{G}_m^m \backslash J^{-1}(m)$, whenever a manifold, is symplectic.

3 The symplectic groupoid of a Poisson Lie group

In this section we recall basic results of the theory of Poisson Lie groups, and of the symplectic groupoid integrating them. The presentation follows [12].

A *Poisson Lie group* G is a Poisson manifold and a Lie group whose multiplication map $G \times G \rightarrow G$ is a Poisson map, where $G \times G$ is endowed with the product Poisson structure. In terms of the Poisson bivector field π_G , it means that

$$\pi_G(gh) = l_g \pi_G(h) + r_h \pi_G(g), \quad \forall g, h \in G, \quad (1)$$

where l_g (r_g) stands for the left (right) group multiplication by g , as well as for the induced tangent map. A multivector field satisfying (1) is said to be *multiplicative*.

A left action $\sigma : G \times P \rightarrow P$ of a Poisson Lie Group G on a Poisson manifold P is called a *Poisson action* if σ is a Poisson map, where $G \times P$ is endowed with the product Poisson structure. In terms of the Poisson bivectors π_G of G and π_P of P , σ is a Poisson action if, for any $g \in G$ and $p \in P$, we have that

$$\pi_P(\sigma(g, p)) = g_* \pi_P(p) + p_* \pi_G(g),$$

where $g : P \rightarrow P$, $g(p) = \sigma(g, p)$ and $p : G \rightarrow P$, $p(g) = \sigma(g, p)$.

A *Lie bialgebra* is the couple $(\mathfrak{g}, \mathfrak{g}^*)$, where $\mathfrak{g} = \text{Lie}G$ and its dual \mathfrak{g}^* is a Lie algebra with bracket map $[\cdot, \cdot]_{\mathfrak{g}^*}$ such that $\delta = [\cdot, \cdot]_{\mathfrak{g}^*}^* : \mathfrak{g} \rightarrow \Lambda^2 \mathfrak{g}$ is a 1-cocycle on \mathfrak{g} relative to the adjoint representation of \mathfrak{g} on $\Lambda^2 \mathfrak{g}$.

Let us assume that the group G is connected and simply connected.

Theorem 1. *There is a one to one correspondence between connected and simply connected Poisson Lie groups and Lie bialgebras.*

Definition 2. *The **double Lie algebra** $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is defined as the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ endowed with the unique Lie bracket structure such that*

- i) it restricts to the given Lie brackets on \mathfrak{g} and \mathfrak{g}^* ;*
- ii) the symmetric and non-degenerate scalar product on $\mathfrak{g} \oplus \mathfrak{g}^*$ defined by*

$$\langle X + \xi, Y + \eta \rangle = \xi(Y) + \eta(X), \quad \forall X, Y \in \mathfrak{g}, \quad \forall \xi, \eta \in \mathfrak{g}^*$$

is invariant.

In particular the bracket is defined for any $X, Y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$ as

$$[X + \xi, Y + \eta] = [X, Y] - ad_\eta^*(X) + ad_\xi^*(Y) + [\xi, \eta] + ad_X^*(\eta) - ad_Y^*(\xi). \quad (2)$$

It can be shown that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ equipped with the bracket (2) is a Lie algebra if and only if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra. As a consequence we have that if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then so is $(\mathfrak{g}^*, \mathfrak{g})$. In particular the connected and simply connected group G^* integrating \mathfrak{g}^* is a Poisson-Lie group. We call it the *dual Poisson Lie group* of (G, π_G) .

The *double Lie group* D is defined as the connected and simply-connected Lie group with Lie algebra \mathfrak{d} . Let $\phi_1 : G \rightarrow D$ and $\phi_2 : G^* \rightarrow D$ be the Lie group homomorphisms obtained by respectively integrating the inclusion maps $\mathfrak{g} \hookrightarrow \mathfrak{d}$ and $\mathfrak{g}^* \hookrightarrow \mathfrak{d}$. In the following we denote $\phi_1(g) = \bar{g}$ and $\phi_2(\gamma) = \bar{\gamma}$.

The following formulas, proved in [12], describe the Poisson tensor of G and G^* in terms of the group structure of D , making explicit the correspondence described in the integration Theorem 1. Indeed, let $p_{\mathfrak{g}} : \mathfrak{d} \rightarrow \mathfrak{g}$, $p_{\mathfrak{g}^*} : \mathfrak{d} \rightarrow \mathfrak{g}^*$ be the natural projections; for any $g \in G$, $\gamma \in G^*$, $X_i \in \mathfrak{g}$ and $\xi_i \in \mathfrak{g}^*$ we have that

$$\begin{aligned} \langle r_{g^{-1}}\pi_G(g), \xi_1 \wedge \xi_2 \rangle &= -\langle p_{\mathfrak{g}}\text{Ad}_{\bar{g}^{-1}}\xi_1, p_{\mathfrak{g}^*}\text{Ad}_{\bar{g}^{-1}}\xi_2 \rangle \quad g \in G, \xi_i \in \mathfrak{g}^*, \\ \langle r_{\gamma^{-1}}\pi_{G^*}(\gamma), X_1 \wedge X_2 \rangle &= \langle p_{\mathfrak{g}}\text{Ad}_{\bar{\gamma}^{-1}}X_1, p_{\mathfrak{g}^*}\text{Ad}_{\bar{\gamma}^{-1}}X_2 \rangle \quad \gamma \in G^*, X_i \in \mathfrak{g}, \end{aligned} \quad (3)$$

where Ad is the adjoint action of D .

For further purposes, let us consider the Poisson tensor π_+ on D defined as follows:

$$\pi_+(d) = \frac{1}{2}(r_d\pi_0 + l_d\pi_0), \quad d \in D,$$

where $\pi_0 \in \mathfrak{d} \wedge \mathfrak{d}$ is defined by $\pi_0(\xi_1 + X_1, \xi_2 + X_2) = \langle X_1, \xi_2 \rangle - \langle X_2, \xi_1 \rangle$, for $\xi_1 + X_1, \xi_2 + X_2 \in \mathfrak{d}^* \cong \mathfrak{g}^* \oplus \mathfrak{g}$. If $d \in D$ can be factorized as $d = \bar{g}\bar{\gamma}$ for some $g \in G$ and $\gamma \in G^*$, then an explicit formula for π_+ is given by

$$\begin{aligned} &\langle (l_{\bar{g}^{-1}}r_{\bar{\gamma}^{-1}})\pi_+(d), (\xi_1 + X_1) \wedge (\xi_2 + X_2) \rangle = \\ &= \langle X_1, \xi_2 \rangle - \langle X_2, \xi_1 \rangle + \langle l_{g^{-1}}\pi_G(g), \xi_1 \wedge \xi_2 \rangle + \langle r_{\gamma^{-1}}\pi_{G^*}(\gamma), X_1 \wedge X_2 \rangle = \\ &= \langle X_1, \xi_2 + \text{Ad}_{\bar{\gamma}}p_{\mathfrak{g}^*}\text{Ad}_{\bar{\gamma}^{-1}}X_2 \rangle - \langle \xi_1, X_2 + \text{Ad}_{\bar{g}^{-1}}p_{\mathfrak{g}}\text{Ad}_{\bar{g}}\xi_2 \rangle. \end{aligned} \quad (4)$$

It can be proved that $\pi_+(g\gamma)$ is nondegenerate.

If $\phi_1 \times \phi_2$ is a global diffeomorphism, then we can identify D with $G \times G^*$ and π_+ defines a symplectic structure on the double. Moreover the global decomposition of D defines a left action of G on G^* and a right action of G^* on G . Let $g \in G$ and $\gamma \in G^*$ and let $g\gamma = {}^g\gamma g^\gamma$, where we identify g with \bar{g} and γ with $\bar{\gamma}$. It is immediate to verify that $(g, \gamma) \rightarrow {}^g\gamma$ is a left action of G on G^* and $(g, \gamma) \rightarrow g^\gamma$ is right action of G^* on G .

These are known as *dressing actions*. It can be easily verified that for any $g, g_i \in G$ and $\gamma, \gamma_i \in G^*$ we have that

$$(g_1 g_2)^\gamma = g_1^{g_2 \gamma} g_2^\gamma ; \quad {}^g(\gamma_1 \gamma_2) = {}^g \gamma_1 \quad {}^{g \gamma_1} \gamma_2 . \quad (5)$$

Such *intertwining* property between the two actions defines what is called a *matched pair of Lie groups* [17, 15]; we will come back to this notion in Section 4.3.

Lemma 3. *The fundamental vector fields associated to the left dressing action of G on G^* and to the right dressing action of G^* on G are respectively:*

$$\mathcal{S}_X(\gamma) = \pi_{G^*}^\#(r_{\gamma^{-1}}^* X) , \quad \forall \gamma \in G^*, \quad X \in \mathfrak{g} \equiv (\mathfrak{g}^*)^* ; \quad (6)$$

$$\mathcal{S}_\xi(g) = -\pi_G^\#(l_{g^{-1}}^* \xi) , \quad \forall g \in G, \quad \xi \in \mathfrak{g}^* .$$

Proof. A direct computation gives the following expressions for the fundamental vector fields associated with the left dressing action of G on G^* and with the right dressing action of G^* on G :

$$\mathcal{S}_X(\gamma) = l_\gamma p_{\mathfrak{g}^*}(\text{Ad}_{\overline{\gamma}^{-1}} X) , \quad \forall \gamma \in G^*, \quad X \in \mathfrak{g};$$

$$\mathcal{S}_\xi(g) = r_g p_{\mathfrak{g}}(\text{Ad}_{\overline{g}} \xi) , \quad \forall g \in G, \quad \xi \in \mathfrak{g}^* .$$

We have to prove that the pointwise pairing of these vector fields with generic 1-forms coincide. Then, given $X, Y \in \mathfrak{g}, \gamma \in G^*$,

$$\begin{aligned} \langle \mathcal{S}_X(\gamma), r_{\gamma^{-1}}^* Y \rangle &= \langle l_\gamma p_{\mathfrak{g}^*}(\text{Ad}_{\overline{\gamma}^{-1}} X), r_{\gamma^{-1}}^* Y \rangle \\ &= \langle p_{\mathfrak{g}^*}(\text{Ad}_{\overline{\gamma}^{-1}} X), \text{Ad}_{\overline{\gamma}^{-1}}^* Y \rangle = \langle p_{\mathfrak{g}^*} \text{Ad}_{\overline{\gamma}^{-1}} X, p_{\mathfrak{g}} \text{Ad}_{\overline{\gamma}^{-1}} Y \rangle \\ &= \langle \pi_{G^*}(\gamma), r_{\gamma^{-1}}^*(Y \wedge X) \rangle \equiv \langle \pi_{G^*}^\#(r_{\gamma^{-1}}^* X), r_{\gamma^{-1}}^* Y \rangle . \end{aligned}$$

The proof for $\mathcal{S}_\xi(g)$ is similar. □

The vector fields (6) are called *dressing vector fields*; their definition depends only on the infinitesimal Lie bialgebra. Therefore they are defined even when $\phi_1 \times \phi_2$ is not a diffeomorphism (and more generally even if ϕ_1, ϕ_2 does not exist). We saw that if $D = G \times G^*$, then the dressing vector fields are complete. In [12], Lu has proved that *i*) the dressing vector fields of G are complete if and only if those of G^* are complete; *ii*) $D = G \times G^*$ if and only if the dressing vector fields are complete.

Integrability of Poisson Lie groups has been shown in [14]. Let us consider the submanifold of $G \times G^* \times G^* \times G$ of dimension $2 \dim G$ defined by

$$\Omega = \{(g_1, \gamma_1, \gamma_2, g_2) \in G \times G^* \times G^* \times G, \quad \overline{g}_1 \overline{\gamma}_1 = \overline{\gamma}_2 \overline{g}_2 \in D\} . \quad (7)$$

The local diffeomorphism $\Phi : \Omega \rightarrow D$, defined as $\Phi(g_1, \gamma_1, \gamma_2, g_2) = \overline{g}_1 \overline{\gamma}_1$, induces a nondegenerate Poisson structure on Ω , that we still denote with π_+ .

Proposition 4. *Let G be a connected and simply connected Poisson Lie group and let G^* be the dual Poisson Lie group. Consider the groupoid $\mathcal{G}(G) = (\Omega, \alpha_G, \beta_G, m_G, \epsilon_G, i_G)$ over G with structure maps:*

- i) $\alpha_G(g_1, \gamma_1, \gamma_2, g_2) = g_1;$
- ii) $\beta_G(g_1, \gamma_1, \gamma_2, g_2) = g_2;$
- iii) $\epsilon_G(g) = (g, e, e, g);$
- iv) $m_G[(g_1, \gamma_1, \gamma_2, g_2)(g_2, \lambda_1, \lambda_2, k_2)] = (g_1, \gamma_1 \lambda_1, \gamma_2 \lambda_2, k_2);$
- v) $\iota_G(g_1, \gamma_1, \gamma_2, g_2) = (g_2, \gamma_1^{-1}, \gamma_2^{-1}, g_1).$

Then $\mathcal{G}(G)$ equipped with π_+^{-1} is a symplectic groupoid integrating (G, π_G) .

Consider the groupoid $\mathcal{G}(G^*) = (\Omega, \alpha_{G^*}, \beta_{G^*}, m_{G^*}, \epsilon_{G^*}, i_{G^*})$ over G^* , with structure maps:

- i) $\alpha_{G^*}(g_1, \gamma_1, \gamma_2, g_2) = \gamma_2;$
- ii) $\beta_{G^*}(g_1, \gamma_1, \gamma_2, g_2) = \gamma_1;$
- iii) $\epsilon_{G^*}(\gamma) = (e, \gamma, \gamma, e);$
- iv) $m_{G^*}[(g_1, \gamma_1, \gamma_2, g_2)(k_1, \lambda_1, \gamma_1, k_2)] = (g_1 k_1, \lambda_1, \gamma_2, g_2 k_2);$
- v) $\iota_{G^*}(g_1, \gamma_1, \gamma_2, g_2) = (g_1^{-1}, \gamma_2, \gamma_1, g_2^{-1}).$

Then $\mathcal{G}(G^*)$ equipped with $-\pi_+^{-1}$ is a symplectic groupoid integrating (G^*, π_{G^*}) .

If G and G^* are complete, then $\Omega = G \times G^*$ globally and the above description can be given in terms of the dressing transformations. In particular the groupoid structures for $\mathcal{G}(G)$ read as $\alpha_G(g\gamma) = g$, $\beta_G(g\gamma) = g^\gamma$, $m_G[(g_1\gamma_1)(g_1^{\gamma_1}\gamma_2)] = (g_1\gamma_1\gamma_2)$, $\epsilon_G(g) = (ge)$, $\iota_G(g\gamma) = g^\gamma\gamma^{-1}$. For $\mathcal{G}(G^*)$ we have $\alpha_{G^*}(g\gamma) = {}^g\gamma$, $\beta_{G^*}(g\gamma) = \gamma$, $m_{G^*}[(g_1{}^{g_2}\gamma_2)(g_2\gamma_2)] = (g_1g_2\gamma_2)$, $\epsilon_{G^*}(\gamma) = (e\gamma)$, $\iota_{G^*}(g\gamma) = g^{-1g}\gamma$.

3.1 The non simply connected case

Let us remove in this subsection the hypothesis that G is simply connected. The above construction of the symplectic groupoid cannot be repeated since now $\phi_1 : \tilde{G} \rightarrow D$, where \tilde{G} is the universal covering of G . Let $Z \subset \tilde{G}$ be the discrete central subgroup such that $G = \tilde{G}/Z$. There exists on \tilde{G} a unique multiplicative Poisson structure $\pi_{\tilde{G}}$ such that the quotient $\tilde{G} \rightarrow G$ is a Poisson map and $\pi_{\tilde{G}}(z) = 0$ for any $z \in Z$. As a consequence the

multiplication by z on \tilde{G} is a Poisson diffeomorphism; moreover by looking at (3) we see that since $\pi_{\tilde{G}}(z) = 0$ we have $\text{Ad}_{\bar{z}}\xi = \text{Ad}_z^*\xi$, for any $\xi \in \mathfrak{g}^*$, and $\text{Ad}_z^*\xi = \xi$ since Z is central. So we can conclude that $\phi_1(z) = \bar{z}$ commutes with $\bar{\gamma}$ for any $\gamma \in G^*$ and that Z acts as a symplectic groupoid morphism on the symplectic groupoid $\mathcal{G}(\tilde{G})$ defined in Proposition 4 as $z(\tilde{g}_1, \gamma_1, \gamma_2, \tilde{g}_2) = (z\tilde{g}_1, \gamma_1, \gamma_2, z\tilde{g}_2)$.

Proposition 5. *For any Poisson Lie group $G = \tilde{G}/Z$, $\mathcal{G}(G) = \mathcal{G}(\tilde{G})/Z$ carries the structure of a symplectic groupoid integrating it.*

In the following we will denote the equivalence classes as $[\tilde{g}_1, \gamma_1, \gamma_2, \tilde{g}_2] \in \mathcal{G}(G)$. Remark that it can happen that $\phi_1 : \tilde{G} \rightarrow D$ satisfies $\phi_1(Z) = 1$ so that it descends to G . In this case it is possible to define a groupoid as in Proposition 4, even without assuming that G is simply connected. It is easily observed that such groupoid is a quotient by Z of the groupoid defined in Proposition 5.

As a simple consequence of Proposition 4, we have the following corollary.

Corollary 6. *The symplectic groupoid $\mathcal{G}(G^*)$ acts symplectically on $\overline{\mathcal{G}(G)}$ with anchor $J : \mathcal{G}(G) \rightarrow G^*$ defined as $J[\tilde{g}_1, \gamma_1, \gamma_2, \tilde{g}_2] = \alpha_{G^*}(\tilde{g}_1, \gamma_1, \gamma_2, \tilde{g}_2) = \gamma_2$; the action $a : \mathcal{G}(G^*)_{\beta_{G^*}} \times_J \mathcal{G}(G) \rightarrow \mathcal{G}(G)$ is given by*

$$a\{(\tilde{k}_1, \lambda_1, \lambda_2, \tilde{k}_2)[\tilde{g}_1, \gamma_1, \lambda_1, \tilde{g}_2]\} = [\tilde{k}_1\tilde{g}_1, \gamma_1, \lambda_2, \tilde{k}_2\tilde{g}_2] .$$

Proof. Simply observe that the graph of the action a is the quotient under the action of Z of the graph of the multiplication of $\mathcal{G}(G^*)$. \square

In particular we have that $J : (\mathcal{G}(G), \pi_+) \rightarrow (G^*, \pi_{G^*})$ is an anti-Poisson map.

4 The symplectic groupoid of a homogeneous space

In this section we discuss the integration of Poisson homogeneous spaces of the Poisson Lie group G .

Let us start with the simplest case of a Lie group G with the zero Poisson structure $\pi_G = 0$. Its symplectic groupoid $\mathcal{G}(G) = G \times \mathfrak{g}^*$ is identified with the cotangent bundle after trivializing via left translations. The left multiplication of G on itself admits a cotangent lift $k(g, \xi) = (kg, \xi)$ for $k, g \in G$ and $\xi \in \mathfrak{g}^*$; this lifted action is hamiltonian with momentum map $J(g, \xi) = \text{Ad}_g^*(\xi)$. Let now H be any closed subgroup of G with Lie algebra \mathfrak{h} . The restriction to H of the lifted action is again obviously hamiltonian with momentum map $J_H = \text{pr}_H \circ J$, where $\text{pr}_H : \mathfrak{g}^* \rightarrow \mathfrak{g}^*/\mathfrak{h}^\perp$ and $\mathfrak{h}^\perp = \{\xi \in \mathfrak{g}^* : \langle \xi, X \rangle = 0, \forall X \in \mathfrak{h}\}$.

The symplectic groupoid of the quotient is just the Marsden–Weinstein reduction of this hamiltonian action: $T^*(H \backslash G) = H \backslash J_H^{-1}(0)$.

We are going to see how this generalizes to a generic Poisson Lie group. We know that any action by Poisson diffeomorphism on an integrable Poisson manifold can be lifted to a hamiltonian action on the (ssc) symplectic groupoid with a multiplicative momentum map, see [18, 7]. This construction applies only to the case H being a Poisson subgroup with zero Poisson structure; it is clear that in the general case one has to consider generalized notions of hamiltonian actions and even in the generalized setting we will consider, the lifting property will not be automatic.

4.1 Embeddable homogeneous spaces

Let us recall some basic facts about coisotropic subgroups and their role in the quotient of Poisson manifolds. Let H be a coisotropic connected closed subgroup of the Poisson Lie group G and $\mathfrak{h} = \text{Lie } H$. At the infinitesimal level, coisotropic subgroups are characterized by the following Proposition, whose proof can be found in [20].

Proposition 7. *A subgroup H is coisotropic if and only if $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is a subalgebra of \mathfrak{g}^* , where $\mathfrak{h}^\perp = \{\xi \in \mathfrak{g}^* : \langle \xi, X \rangle = 0, \forall X \in \mathfrak{h}\}$.*

Assumption: Let \mathfrak{h}^\perp be integrated by a closed subgroup $H^\perp \subset G^*$.

As a consequence of Proposition 7, H^\perp results coisotropic as well. The following property of coisotropic subgroups will be relevant in what follows.

Lemma 8. *Given a coisotropic subgroup $H \subset G$, then the restriction of the (infinitesimal) dressing actions of G on G^* to H leaves H^\perp invariant and its orbits are the coisotropic leaves. Moreover, when H is a Poisson subgroup, then the dressing action of H^\perp on H is trivial.*

Proof. Since for any $\gamma \in H^\perp$ we can characterize $T_\gamma H^\perp$ as $r_\gamma \mathfrak{h}^\perp$, we get that $N_\gamma^* H^\perp = r_{\gamma^{-1}}^* \mathfrak{h}$. Then the dressing vector fields $\mathcal{S}_X(\gamma) = \pi_{G^*}^\sharp(r_{\gamma^{-1}}^* X)$ corresponding to $X \in \mathfrak{h}$ (see Lemma 3) span the coisotropic distribution and, in particular, are tangent to H^\perp . Analogously for the right dressing action. To prove the last statement, let us recall that the coisotropic foliation of a Poisson submanifold is null. \square

Corollary 9. *If H is a Poisson subgroup, then H acts on H^\perp by automorphisms.*

Proof. Let us prove it in the complete case. Let $h \in H$, $\gamma_i \in H^\perp$. By the properties (5) of the dressing action, ${}^h(\gamma_1 \gamma_2) = {}^h \gamma_1 \cdot {}^{h\gamma_1} \gamma_2$. As $h\gamma_1 = h$ by means of Lemma 8, then ${}^h(\gamma_1 \gamma_2) = {}^h \gamma_1 \cdot {}^h \gamma_2$. In the general case, infinitesimal action by automorphisms means that the dressing vector fields of H on H^\perp are multiplicative. \square

Let us denote with \tilde{H} , \tilde{G} the universal covers of H and G respectively and let $\phi_{\tilde{H}} : \tilde{H} \rightarrow \tilde{G}$ the group homomorphism integrating the inclusion $\mathfrak{h} \rightarrow \mathfrak{g}$. Then let us define

$$\Omega(\tilde{H}, H^\perp) = \{(\phi_{\tilde{H}}(\tilde{h}_1), \gamma_1, \gamma_2, \phi_{\tilde{H}}(\tilde{h}_2)) \in \mathcal{G}(\tilde{G}), \tilde{h}_i \in \tilde{H}, \gamma_i \in H^\perp\} \subset \Omega. \quad (8)$$

It is clear that $\Omega(\tilde{H}, H^\perp)$ defines a subgroupoid $\mathcal{G}(\tilde{G}, \phi_{\tilde{H}}(\tilde{H}))$ over $\phi_{\tilde{H}}(\tilde{H})$ of $\mathcal{G}(\tilde{G})$ and a subgroupoid $\mathcal{G}(G^*, H^\perp)$ over H^\perp of $\mathcal{G}(G^*)$. They respectively integrate the subalgebroids $N^*\phi_{\tilde{H}}(\tilde{H}) \subset T^*\tilde{G}$ and $N^*H^\perp \subset T^*G^*$.

The following theorem establishes the role of coisotropic subgroups in the quotient of Poisson manifolds (the proof can be found in [12]).

Theorem 10. *Let $\sigma : K \times P \rightarrow P$ be a Poisson action of the Poisson Lie group K over the Poisson manifold P and let $B \subset K$ be a coisotropic subgroup of K . If the orbit space $B \backslash P$ is a smooth manifold, then there is a unique Poisson structure on $B \backslash P$ such that the natural projection map $P \rightarrow B \backslash P$ is Poisson.*

Let H be a coisotropic subgroup of the Poisson Lie group G ; if we apply this result to $P = K = G$, $B = H$ and to $P = K = G^*$, $B = H^\perp$ we conclude that both $H \backslash G$ and G^*/H^\perp are Poisson manifolds. Borrowing the terminology from quantum groups we call them *embeddable Poisson homogeneous spaces*, since H -invariant functions on G are a Poisson subalgebra of $C^\infty(G)$. Let $p_H : G \rightarrow H \backslash G$ and $p_{H^\perp} : G^* \rightarrow G^*/H^\perp$ the projection maps and let us denote $p_H(g) = \underline{g}$ and $p_{H^\perp}(\gamma) = \underline{\gamma}$. Embeddable Poisson homogeneous spaces come with a distinct point, the image \underline{e} of the identity e , where the coinduced Poisson structure vanishes. Indeed, they can be characterized as those having at least one point where the Poisson structure vanishes, or equivalently the stability group of such a point is coisotropic.

4.2 The complete and simply connected case

Let us assume that G is simply connected and complete. Let us review the concept of symplectic reduction via Lu's momentum map.

Definition 11. *A C^∞ map $J : P \rightarrow G^*$ is called a momentum mapping for the left Poisson action $\sigma : G \times P \rightarrow P$ if*

$$\sigma_X = -\pi_P^\#(J^*(X^r)), \quad \forall X \in \mathfrak{g}.$$

where for each $X \in \mathfrak{g}$, X^r is the right invariant 1-form on G^* with value X at e , and σ_X is the fundamental vector field associated to X by the action σ . The momentum mapping J is said to be equivariant if $J(\sigma(g, p)) = {}^g(J(p))$, for any $g \in G$ and $p \in P$.

Remark that this definition is slightly different from the one given in [12]. If P is symplectic, a Poisson action σ admits a momentum mapping J if and only if there exists a symplectic action of the symplectic groupoid $\mathcal{G}(G^*)$ on P with anchor J : the correspondence is given by $\sigma(g, p) = (gJ(p))p$, for $g \in G$, $p \in P$, see [25]. Remark that this correspondence demands G to be complete. By applying this result to $P = \mathcal{G}(G)$ we get that the left G -action on $\mathcal{G}(G)$ given by $\sigma(g, (k\gamma)) = gk\gamma$ is Poisson and admits an equivariant momentum mapping $J(g\gamma) = {}^g\gamma$. Moreover it is multiplicative, *i.e.* if $(g_1\gamma_1)$ and $(g_2\gamma_2)$ are composable then $J((g_1\gamma_1)(g_2\gamma_2)) = J(g_1\gamma_1)J(g_2\gamma_2)$. More concretely, for any $X \in \mathfrak{g}$, the fundamental vector field of the left G -action is $\sigma_X(g\gamma) = r_{(g\gamma)^{-1}*}X$ and the right invariant form $X_\gamma^r = r_{\gamma^{-1}}^*X$. We have that, for any $g\gamma \in \mathcal{G}(G)$,

$$\sigma_X(g\gamma) = -\pi_+^\sharp \circ [T_{g\gamma}J]^*(X_{J(g\gamma)}^r). \quad (9)$$

Let us introduce the map

$$J_H = p_{H^\perp} \circ J, \quad J_H(g\gamma) = \underline{{}^g\gamma}. \quad (10)$$

Being a composition of Poisson submersions, J_H is a Poisson submersion too. In the special case in which H is a Poisson Lie subgroup, H^\perp is a normal subgroup and $G^*/H^\perp \equiv H^*$ is a Poisson Lie group with Lie algebra $\mathfrak{h}^* = \mathfrak{g}^*/\mathfrak{h}^\perp$. By using (5) it can be easily shown that the dressing action of H descends to G^*/H^\perp , ${}^h p_{H^\perp}(\gamma) \equiv p_{H^\perp}({}^h\gamma)$, for any $h \in H$ and $\gamma \in G^*$. Then J_H is an equivariant and multiplicative momentum mapping for the left multiplication by H . In the general coisotropic case, G^*/H^\perp is only a Poisson manifold and J_H must be thought as a momentum mapping in a generalized sense, see Corollary 13 at the end of this subsection.

Let us consider $J_H^{-1}(\underline{e}) = \{g\gamma \in G \times G^* : {}^g\gamma \in H^\perp\}$, that is a submanifold since J_H is a submersion. Since $\{\underline{e}\}$ is a zero dimensional leaf of G^*/H^\perp then $J_H^{-1}(\underline{e})$ is coisotropic in $G \times G^*$. Let us show that the left multiplication by H leaves $J_H^{-1}(\underline{e})$ invariant: Lemma 8 implies that, if $g\gamma \in J_H^{-1}(\underline{e})$, that is if ${}^g\gamma \in H^\perp$, then ${}^{hg}\gamma \equiv {}^h({}^g\gamma) \in {}^h(H^\perp) = H^\perp$, and then $hg\gamma \in J_H^{-1}(\underline{e})$. The left H action is proper and free so that the orbit space is smooth.

Theorem 12. $\mathcal{G}(H \backslash G) = H \backslash J_H^{-1}(\underline{e})$ is a symplectic groupoid that integrates $H \backslash G$.

Proof. Since $J_H^{-1}(\underline{e})$ is a coisotropic submanifold of $\mathcal{G}(G)$, it admits a symplectic reduction and let ω_+^H the induced symplectic structure. Let us show that this reduction coincides with the quotient by the left H action. Formula 9 implies that the fundamental vector fields associated to the left multiplication of $G \times G^*$ by H are given by

$$\sigma_X(g\gamma) = -\pi_+^\sharp \left[[T_{g\gamma}J]^* \circ r_{J(g\gamma)^{-1}}^* X \right], \quad \forall X \in \mathfrak{h}, \quad (11)$$

where we recall that $J : \mathcal{G}(G) \rightarrow G^* : g\gamma \mapsto {}^g\gamma$ is the momentum mapping. On the other hand, the coisotropic distribution is defined as

$$\pi_+^\sharp (N_{g\gamma}^* J_H^{-1}(\underline{e})) , \quad \forall g\gamma \in J_H^{-1}(\underline{e}).$$

Simple identities allow to write for $g\gamma \in J_H^{-1}(\underline{e})$

$$\begin{aligned} N_{g\gamma}^* J_H^{-1}(\underline{e}) &= \text{Ker}[T_{g\gamma} J_H]^\perp = \text{Im}[T_{g\gamma} J_H]^* = \{[T_{g\gamma} J]^* \circ [T_{J(g\gamma)} p_{H^\perp}]^* X, \quad \forall X \in \mathfrak{h}\} \\ &= \{[T_{g\gamma} J]^* \circ r_{J(g\gamma)^{-1}}^* X, \quad \forall X \in \mathfrak{h}\}, \end{aligned}$$

since $p_{H^\perp} \circ r_{J(g\gamma)^{-1}} = p_{H^\perp}$ and $[T_e p_{H^\perp}]^* : T_{\underline{e}}^*(G^*/H^\perp) \equiv \mathfrak{h} \rightarrow T_e^* G^* \equiv \mathfrak{g}$ is just the inclusion map.

Let us show that $J_H^{-1}(\underline{e})$ is a subgroupoid of $\mathcal{G}(G)$. Take $g_1\gamma_1, g_2\gamma_2 \in J_H^{-1}(\underline{e})$ such that $g_2 = g_1^{\gamma_1}$. Then $m(g_1\gamma_1, g_2\gamma_2) = g_1\gamma_1\gamma_2 \in J_H^{-1}(\underline{e})$ since ${}^{g_1}(\gamma_1\gamma_2) = {}^{g_1}\gamma_1 {}^{g_1^{\gamma_1}}\gamma_2 \in H^\perp$ because ${}^{g_1^{\gamma_1}}\gamma_2 = {}^{g_2}\gamma_2 \in H^\perp$ by definition. The quotient $H \backslash J_H^{-1}(\underline{e})$ inherits the structure of groupoid. In fact the left H action on $J_H^{-1}(\underline{e})$ and on G defines the relations $S_H \subset J_H^{-1}(\underline{e}) \times J_H^{-1}(\underline{e})$ and $R_H \subset G \times G$, respectively; one can show that (S_H, R_H) is a smooth congruence on $J_H^{-1}(\underline{e})$, according to Definition 2.4.5 of [16], that induces a unique Lie groupoid structure on the quotient. We will follow a more direct way, by explicitly defining this groupoid structure. Let us denote with $\underline{g} \in H \backslash G$ the equivalence class of $g \in G$ and with $\underline{g\gamma}$ the equivalence class of $g\gamma \in J_H^{-1}(\underline{e})$. The source and target maps are defined as

$$\begin{aligned} \alpha_G^H(\underline{g\gamma}) &= \underline{\alpha_G(g\gamma)} = \underline{g} \\ \beta_G^H(\underline{g\gamma}) &= \underline{\beta_G(g\gamma)} = \underline{g^\gamma}; \end{aligned}$$

one must check that the definition of β_G^H is correct; indeed $\beta_G^H(hg\gamma) = \underline{(hg)^\gamma} = \underline{h^{g\gamma} g^\gamma} = \underline{g^\gamma}$ since ${}^hg\gamma \in H^\perp$. Given $\underline{g_i\gamma_i} \in H \backslash J_H^{-1}(\underline{e})$, $i = 1, 2$, such as $\alpha_G^H(\underline{g_2\gamma_2}) = \underline{g_2} = \underline{g_1^{\gamma_1}} = \beta_G^H(\underline{g_1\gamma_1})$, we set $m_G^H(\underline{g_1\gamma_1}, \underline{g_2\gamma_2}) = \underline{g_1\gamma_1\gamma_2}$. Then $\alpha_G^H(\underline{g_1\gamma_1\gamma_2}) = \underline{g_1} = \alpha_G^H(\underline{g_1\gamma_1})$ and $\beta_G^H(\underline{g_1\gamma_1\gamma_2}) = \underline{g_1^{\gamma_1\gamma_2}} = \underline{(hg_2)^{\gamma_2}} = \underline{h^{g_2\gamma_2} g_2^{\gamma_2}} = \beta_G^H(\underline{g_2\gamma_2})$, where $h \in H$ is such that $hg_2 = g_1^{\gamma_1}$, which follows from the condition of composability, and last equality follows, once more, from the condition that ${}^{g_2}\gamma_2 \in H^\perp$.

Finally, let us show that the reduced symplectic form ω_+^H is multiplicative. First, we observe that the restriction of $\omega_+ = \pi_+^{-1}$ to $J_H^{-1}(\underline{e})$ is multiplicative, making it a presymplectic groupoid. Then we observe that the quotient map $p_H^{(1)} : J_H^{-1}(\underline{e}) \rightarrow \mathcal{G}(H \backslash G)$ induces a submersion $p_H^{(2)} : J_H^{-1}(\underline{e})^{(2)} \rightarrow \mathcal{G}(H \backslash G)^{(2)}$, so that any element in $\Lambda^2 T\mathcal{G}(H \backslash G)^{(2)}$ can be written as $p_{H*}^{(2)} V$ for $V \in \Lambda^2 T J_H^{-1}(\underline{e})^{(2)}$. We then have

$$\langle (m_G^H - \text{pr}_{G1}^H - \text{pr}_{G2}^H) \omega_+^H, p_{H*}^{(2)} V \rangle = \langle (m^* - \text{pr}_1^* - \text{pr}_2^*) \omega_+, V \rangle = 0.$$

□

One can think of this reduction as a reduction of a symplectic groupoid action. Let $\mathcal{G}(G^*/H^\perp)$ the symplectic groupoid integrating G^*/H^\perp obtained by the right counterpart of the above procedure. In total analogy with the above construction we have that $\mathcal{G}(G^*/H^\perp) = \{g\underline{\gamma} \in G^* \times G^*/H^\perp, g^\gamma \in H\}$. The groupoid structures are $\alpha_{G^*}^{\mathbb{H}^\perp}(g\underline{\gamma}) = \underline{g}\underline{\gamma}$, $\beta_{G^*}^{\mathbb{H}^\perp}(g\underline{\gamma}) = \underline{\gamma}$, $m_{G^*}^{\mathbb{H}^\perp}[(g_1\underline{\gamma}_1)(g_2\underline{\gamma}_2)] = g_1g_2\underline{\gamma}_2$ for $\underline{\gamma}_1 = \underline{g_2}\underline{\gamma}_2$, etc... It is also clear that the isotropy group of \underline{e} is $\mathcal{G}(G^*/H^\perp)_{\underline{e}}^e = H$. One can easily check that $J_H : \mathcal{G}(G) \rightarrow G^*/H^\perp$ is the anchor for the symplectic action of $\mathcal{G}(G^*/H^\perp)$ on $\mathcal{G}(G)$ given by $(k\underline{\lambda})(g\gamma) = kg\gamma$.

Corollary 13. $\mathcal{G}(H \backslash G) = \mathcal{G}(G^*/H^\perp)_{\underline{e}}^e \backslash J_H^{-1}(\underline{e})$.

4.3 The general case

If G is neither complete nor simply connected, we have to use the general form of $\mathcal{G}(G)$ given in Proposition 5. Most of the construction of the complete and simply connected case can be generalized in a straightforward way, apart from few crucial facts. Instead of Lu's momentum map, we have to think of $\mathcal{G}(G)$ as a hamiltonian $\mathcal{G}(G^*)$ space, as described in Corollary 6. In particular, the map J_H defined as in (10) is still a Poisson submersion and $J_H^{-1}(\underline{e})$ is a coisotropic submanifold of $\mathcal{G}(G)$. The coisotropic reduction is therefore well defined. First of all, care has to be taken about smoothness of this quotient. In fact, in the general case the $\mathcal{G}(G^*)$ -action of Corollary 6 does not define a left G -action, due to non completeness of dressing vector fields. In particular, formula (11) still defines an infinitesimal action of \mathfrak{h} , spanning the coisotropic distribution, that cannot in general be integrated to a group action of H .

We first remark that it can be integrated to a groupoid action. In fact the restriction of the groupoid action $a : \mathcal{G}(G^*)_{\beta_{G^*}} \times_J \mathcal{G}(G) \rightarrow \mathcal{G}(G)$ defined in Corollary 6 to the subgroupoid $\mathcal{G}(G^*, H^\perp)$ defined in (8) leaves $J_H^{-1}(\underline{e})$ invariant, *i.e.* we have a left groupoid action $a : \mathcal{G}(G^*, H^\perp)_{\beta_{G^*}} \times_J J_H^{-1}(\underline{e}) \rightarrow J_H^{-1}(\underline{e})$ with anchor $J : J_H^{-1}(\underline{e}) \rightarrow H^\perp$. Furthermore, its infinitesimal action is the restriction of the algebroid action of T^*G^* on $\mathcal{G}(G)$, and by repeating the argument in the proof of Theorem 12, spans the coisotropic distribution of $J_H^{-1}(\underline{e})$. The coisotropic quotient can be obtained as $\mathcal{G}(G^*, H^\perp) \backslash J_H^{-1}(\underline{e})$, the orbit space of the groupoid action of $\mathcal{G}(G^*, H^\perp)$. Although $J_H^{-1}(\underline{e})$ is still a subgroupoid of $\mathcal{G}(G)$, it is not obvious a priori that this groupoid action defines a smooth congruence in $J_H^{-1}(\underline{e})$ and so a groupoid structure on the quotient.

In order to overcome these problems, we introduce a weaker notion of completeness of dressing vector fields reducing the problem to a quotient by an ordinary free and proper group action. We say that two Lie algebras $(\mathfrak{g}_1, \mathfrak{g}_2)$ are a *matched pair of Lie algebras* if there exists a third Lie algebra $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$, called the *double Lie algebra*, isomorphic to $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ as vector space and containing \mathfrak{g}_1 and \mathfrak{g}_2 as Lie subalgebras. It comes out that a and b

defined by $[X, \xi] = b_\xi(X) \oplus a_X(\xi)$ for $X \in \mathfrak{g}_1$, $\xi \in \mathfrak{g}_2$, define compatible actions of the two Lie algebras on each other. Accordingly, we say that two Lie groups G_1 and G_2 form a *matched pair of Lie groups* if there exists a third Lie group $G_1 \bowtie G_2$, called the *double Lie group*, diffeomorphic to $G_1 \times G_2$ and containing G_1 and G_2 as closed Lie subgroups. It comes out that G_1 and G_2 act on each other with actions satisfying compatibility conditions analogue to (5); viceversa, given such compatible actions then there exists a unique double Lie group (see [17], [15]).

It is clear that the couples $(\mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{h}, \mathfrak{h}^\perp)$ form matched pairs of Lie algebras with the coadjoint actions as compatible actions. We say that (H, H^\perp) are *relatively complete* if the infinitesimal actions of \mathfrak{h} on H^\perp and of \mathfrak{h}^\perp on H via the dressing vector fields can be integrated in such a way that (H, H^\perp) forms a matched pair of Lie groups, or equivalently that the finite actions of H and H^\perp satisfy relations (5). We need the following Lemma concerning the universal cover of H , in order to include the case when G is not simply connected.

Lemma 14. *i) If (H, H^\perp) forms a matched pair of Lie groups then also (\tilde{H}, H^\perp) forms a matched pair, where \tilde{H} is the universal cover of $H = \tilde{H}/Z_H$.*

ii) The center $Z_H \subset \tilde{H}$ acts trivially on H^\perp and is a fixed point set of the dressing action of H^\perp .

iii) The quotient map $\tilde{H} \rightarrow H$ intertwines the H^\perp action, i.e. $\tilde{h}^\gamma \rightarrow h^\gamma$ for any lift \tilde{h} of $h \in H$ and $\gamma \in H^\perp$.

Proof. Let us prove point i). Any $\tilde{h} \in \tilde{H}$ can be seen as the equivalence class $[h]$ of a path $h : [0, 1] \rightarrow H$, with $h(0) = e$, with respect to homotopies preserving end points. Since H^\perp is connected and its action on H preserves the identity, for any $\gamma \in H^\perp$ and $[h] \in \tilde{H}$, define the action of H^\perp on \tilde{H} as $[h]^\gamma = [h^\gamma]$, where $h^\gamma(t) = h(t)^\gamma$, and the action of \tilde{H} on H^\perp as $^{[h]}\gamma = h^{(1)}\gamma$. It is immediate to check that they are well defined and that $^{[h]}(\gamma_1\gamma_2) = ^{[h]}\gamma_1 ^{[h]\gamma_1}\gamma_2$. In order to check that $([h_1][h_2])^\gamma = [h_1]^{[h_2]}\gamma [h_2]^\gamma$ we have to prove that the paths $t \rightarrow h_1(t)^{h_2(t)\gamma}$ and $t \rightarrow h_1(t)^{h_2(1)\gamma}$ are homotopic. This can be shown by using the homotopy $F(s, t) = h_1(t)^{h_2(t+s(1-t))\gamma}$.

In order to get point ii), we observe that $Z_H = \pi_1(H)$ is realized as homotopy classes of loops and its action on H^\perp is trivial; the action of H^\perp on Z_H is trivial since H^\perp is connected to the identity so that any loop z is homotopic to z^γ . Finally, point iii) can be directly verified once one realizes that the quotient $\tilde{H} \rightarrow H$ is realized as $[h] \rightarrow h(1)$. \square

Let $\tilde{H} \bowtie H^\perp$ be the double Lie group with the product rule given by $\tilde{h}\gamma = \tilde{h}\gamma\tilde{h}^\gamma$, for $\tilde{h} \in \tilde{H}$ and $\gamma \in H^\perp$.

Theorem 15. *If (H, H^\perp) are relatively complete then the left groupoid action $a : \mathcal{G}(G^*, H^\perp)_{\beta_{G^*} \times_J J_H^{-1}(\underline{e})} \rightarrow J_H^{-1}(\underline{e})$ is equivalent to a free and proper action of H and $H \backslash J_H^{-1}(\underline{e})$ is a symplectic groupoid that integrates $H \backslash G$.*

Proof. Let \tilde{G} be the universal covering of G so that $G = \tilde{G}/Z$ and let $\mathcal{G}(G) = Z \backslash \mathcal{G}(\tilde{G})$ be as described in Proposition 5. Let us recall that $\phi_1 : \tilde{G} \rightarrow D$ and $\phi_2 : G^* \rightarrow D$ are the Lie group homomorphisms entering the definition of $\mathcal{G}(\tilde{G})$.

We have that $[\tilde{g}_1, \gamma_1, \gamma_2, \tilde{g}_2] \in J_H^{-1}(\underline{e})$ if $\gamma_2 \in H^\perp$. Let $\phi_{\tilde{H}} : \tilde{H} \rightarrow \tilde{G}$ be the Lie group homomorphism induced by the injection $\mathfrak{h} \rightarrow \mathfrak{g}$. Then we have that $\phi_{\tilde{H}}(Z_H) \subset Z$. Moreover, due to the uniqueness of the group homomorphism integrating any Lie algebra morphism, we can conclude that $\psi : \tilde{H} \bowtie H^\perp \rightarrow D$ defined as $\psi(\tilde{h}\gamma) = \phi_1(\phi_{\tilde{H}}(\tilde{h}))\phi_2(\gamma)$, for $\tilde{h} \in \tilde{H}$ and $\gamma \in H^\perp$, is a group homomorphism. For any $\tilde{h} \in \tilde{H}$, $\lambda \in H^\perp$ we have that $(\phi_{\tilde{H}}(\tilde{h}), \lambda, {}^{\tilde{h}}\lambda, \phi_{\tilde{H}}(\tilde{h}^\lambda)) \in \mathcal{G}(G^*, H^\perp)$. In fact, by using the definition above of ψ we have

$$\begin{aligned} \phi_1(\phi_{\tilde{H}}(\tilde{h}))\phi_2(\lambda) &= \psi(\tilde{h}\lambda) = \psi({}^{\tilde{h}}\lambda\tilde{h}^\lambda) = \psi({}^{\tilde{h}}\lambda)\psi(\tilde{h}^\lambda) = \\ &= \phi_2({}^{\tilde{h}}\lambda)\phi_1(\phi_{\tilde{H}}(\tilde{h}^\lambda)) . \end{aligned}$$

So we can define the left H -action on $J_H^{-1}(\underline{e})$ by choosing any lift \tilde{h} of $h \in H$ and letting

$$h[\tilde{g}_1, \gamma_1, \gamma_2, \tilde{g}_2] = a\{(\phi_{\tilde{H}}(\tilde{h}), \gamma_2, {}^{\tilde{h}}\gamma_2, \phi_{\tilde{H}}(\tilde{h}^\gamma))[\tilde{g}_1, \gamma_1, \gamma_2, \tilde{g}_2]\} = [\phi_{\tilde{H}}(\tilde{h})\tilde{g}_1, \gamma_1, {}^{\tilde{h}}\gamma_2, \phi_{\tilde{H}}(\tilde{h}^{\gamma_2})\tilde{g}_2] .$$

The independence on the choice of the lift \tilde{h} is clear since $(\tilde{h}z)^\gamma = \tilde{h}^\gamma z$ for any $z \in Z_H$ due to point *ii*) in Lemma 14.

Under this condition, the coisotropic reduction is obtained as a quotient of the free and proper action of H and so it is a smooth manifold. Moreover, groupoid structures descend to the quotient, as it can be directly verified and everything goes through like in the proof of Theorem 12. \square

In the following we analyze some obvious conditions that imply relative completeness.

Lemma 16. *If H^\perp is simply connected and H is a Poisson-Lie subgroup, then (H, H^\perp) are relatively complete.*

Proof. Due to Lemma 8, the dressing vector fields of H^\perp restricted to H are zero and the action of H^\perp is trivially integrated. Since the hypothesis *ii*) of Lemma 4.1 in [17] is obviously satisfied and H^\perp is simply connected, we get the result. \square

4.4 An Example: $G = SU(1, 1)$, $H = U(1)$.

Let us consider the following double Lie algebra $\mathfrak{d} = sl(2, \mathbb{C})$ with pairing $\langle A, B \rangle = \text{Im Tr}(AB)$ and

$$\begin{aligned}\mathfrak{g} &= \mathfrak{su}(1, 1) = \left\{ \begin{pmatrix} ia & b \\ b^* & -ia \end{pmatrix}, a \in \mathbb{R}, b \in \mathbb{C} \right\}, \\ \mathfrak{g}^* &= \mathfrak{sb}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & n \\ 0 & -a \end{pmatrix}, a \in \mathbb{R}, n \in \mathbb{C} \right\}.\end{aligned}$$

Since the group

$$G = SU(1, 1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, |\alpha|^2 - |\beta|^2 = 1 \right\}$$

is embedded in $D = SL(2, \mathbb{C})$, formulas (3) define a multiplicative Poisson structure on $SU(1, 1)$, even if it is not simply connected. The simply connected dual group is

$$G^* = SB(2, \mathbb{C}) = \left\{ \begin{pmatrix} A & N \\ 0 & A^{-1} \end{pmatrix}, A > 0, N \in \mathbb{C} \right\}.$$

Let us choose as subgroup $H \subset G$ the diagonal $U(1)$, which is a Poisson-Lie subgroup; then H^\perp is the closed subgroup of G^* of strictly upper diagonal matrices

$$H^\perp = \left\{ \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}, N \in \mathbb{C} \right\}.$$

The quotient $U(1) \backslash SU(1, 1)$ is homeomorphic to the open disk and its quantization has been studied in [11].

Dressing transformations are not complete. An easy way of looking at it is the following. Let $g \in SU(1, 1)$ and $\xi \in \mathfrak{g}^*$: the flux g_t of the dressing vector field corresponding to ξ is given locally by the solution of $g \exp t\xi = \gamma_t g_t$ with $\gamma_t \in G^*$. We see that for $t \in \mathbb{R}$ the equation

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_t & N_t \\ 0 & A_t^{-1} \end{pmatrix} \begin{pmatrix} \sigma_t & \tau_t \\ \tau_t^* & \sigma_t^* \end{pmatrix}$$

admits in general solutions only for $t < t_0$ (for instance take $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$). In particular the dressing action of H^\perp on G is not complete. On the contrary, one easily computes that

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha^2 b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix},$$

so that (H, H^\perp) are relatively complete. Then Theorem 15 produces a symplectic groupoid for the quotient Poisson structure on the disc. A subfamily of the whole family of covariant Poisson discs given in [10] can be described in a similar manner.

4.5 Comparison with the construction in [13]

In [13] the most general Poisson homogeneous spaces of Poisson Lie groups are considered. Drinfeld in [6] showed that Poisson structures on $H \backslash G$, such that the right G action is Poisson, are naturally associated to lagrangian subalgebras $\mathfrak{l} \subset \mathfrak{d}$. The case of H coisotropic, considered in this paper, corresponds to $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Let us assume that *i)* G is a closed subgroup of any D integrating \mathfrak{d} (even not simply connected); *ii)* $H = L_H \cap G$, where L_H is the connected subgroup of D integrating \mathfrak{l} ; *iii)* the infinitesimal action of \mathfrak{l} on G is integrated to a finite action of L_H . Then a Poisson groupoid for any Poisson homogeneous space, even non embeddable, is constructed. Moreover conditions for the Poisson structure to be non degenerate are given.

If we restrict to the embeddable homogeneous spaces, that we consider in the present paper, and to the complete case, the groupoid is described as $G \times_H H^\perp$, the fibred product with respect to the right H -action on $G \times H^\perp$ given by $(g, \gamma)h = (gh, {}^h\gamma)$. In this case the Poisson structure is non degenerate. We can describe our symplectic groupoid $\mathcal{G}(H \backslash G)$ as a fibred product $H^\perp \times_H G$ with respect to the left action $h(\gamma, g) = ({}^h\gamma, hg)$, via the correspondence $(\underline{g}\gamma) \in J_H^{-1}(\underline{e}) \rightarrow [{}^g\gamma, g] \in H^\perp \times_H G$. It is then clear that the reduction procedure in [13] coincides with the right version of our procedure.

In the non complete case the two constructions are different. The groupoid in [13] is described as $\Gamma = G \times_H L_H/H$, where $L_H \subset D$ is the connected subgroup integrating $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{h}^\perp \subset \mathfrak{d}$ and L_H/H is the homogeneous space by right quotient of $H \subset L_H$. The hypothesis that the action of \mathfrak{l} on G can be integrated implies that the dressing vector fields corresponding to \mathfrak{h}^\perp are complete on G . The Poisson structure is not known to be symplectic in general.

In order to realize that this construction is, in general, different from ours, it is enough to look at the trivial case, where $H = \{e\}$ and $H^\perp = G^*$. This case obviously satisfies the relative completeness requirement: the symplectic groupoid described in Theorem 15 is obviously the unreduced one $\mathcal{G}(G)$. The construction in [13] gives $\Gamma = G \times G^*$, which is not a groupoid. In fact, the requirement that the action of $\mathfrak{l} = \mathfrak{g}^*$ integrates is equivalent to completeness.

A less trivial case is given by the example discussed in subsection 4.4. In that case, $L_H = H \bowtie H^\perp$. We saw in fact that the infinitesimal dressing action of H^\perp is complete only when restricted to H , where it is trivial, and is not complete on the whole $SU(1, 1)$.

5 Concluding remarks

In the complete case, the symplectic groupoid $\mathcal{G}(H \backslash G)$ described in the previous section has the source fibre isomorphic to H^\perp , so it will be the unique source simply connected groupoid integrating $H \backslash G$ only if H^\perp is simply connected. Moreover, since we are interested in the problem of quantization a more explicit description will be necessary. In particular it is natural to ask when it is symplectomorphic to $T^*(H \backslash G)$ with the canonical symplectic form. This problem will be addressed elsewhere, while in this section we will sketch a construction of a diffeomorphism between the symplectic groupoid and the cotangent bundle, that can be considered a first step in this direction.

In the complete case, the symplectic groupoid $\mathcal{G}(H \backslash G)$ can be described as the fibre bundle $H^\perp \times_H G$, associated with the principal bundle $G \rightarrow H \backslash G$ and the dressing action of H on H^\perp . Since the cotangent bundle is the bundle associated to the coadjoint action on \mathfrak{h}^\perp , let us suppose that there exists a diffeomorphism $s_H : \mathfrak{h}^\perp \rightarrow H^\perp$ that intertwines the coadjoint action of H with the dressing action, *i.e.* $s_H(\text{Ad}_h^* \xi) = {}^h s_H(\xi)$, for any $h \in H$, $\xi \in \mathfrak{h}^\perp$. We then have a fibre bundle isomorphism that we describe as follows. Let us consider any trivialization of the principal bundle $G \rightarrow H \backslash G$ given by the local sections $g_i : U_i \rightarrow G$ and transition functions $h_{ij} : U_i \cap U_j \rightarrow H$, such that $g_j(x) = h_{ji}(x)g_i(x)$ for any $x \in U_i \cap U_j$. Then there exist local diffeomorphisms $S_i : \mathcal{G}|_{U_i} \rightarrow T^*H \backslash G|_{U_i}$ given by:

$$S_i(x\gamma) = \text{Ad}_{g_i(x)^{-1}}^*(s_H^{-1}(g_i(x)\gamma)) \in \text{Ad}_{g_i(x)^{-1}}^* \mathfrak{h}^\perp = T_x^*(H \backslash G).$$

Since s_H intertwines coadjoint and dressing action of H , $S_i = S_j$ on $U_i \cap U_j$ so that a global diffeomorphism $S : \mathcal{G}(H \backslash G) \rightarrow T^*(H \backslash G)$ is defined. Since the source map, when transported to the cotangent bundle, coincides with the bundle projection, the symplectic structure cannot be the canonical one, unless the Poisson structure on $H \backslash G$ is trivial.

Let us briefly see a class of examples where to apply the above construction. When H is a Poisson Lie group and H^\perp is of exponential type, *i.e.* $H^\perp = \exp \mathfrak{h}^\perp$, we can choose $s_H = \exp$. In fact, since \mathfrak{h}^\perp is an ideal, the coadjoint action of H on H^\perp is a Lie algebra morphism: $\text{Ad}_X^*([\xi, \eta]) = [\text{Ad}_X^* \xi, \eta] + [\xi, \text{Ad}_X^* \eta]$, for all $X \in \mathfrak{h}, \xi, \eta \in \mathfrak{h}^\perp$. Then due to the uniqueness of the group automorphism that integrates the coadjoint action we conclude that ${}^h \exp \xi = \exp \text{Ad}_h^* \xi$.

While we were finishing this paper, it appeared on the net paper [21] that contains very close results. It is shown that the Poisson action of a complete Poisson Lie group H on an integrable Poisson manifold P can be lifted to a groupoid action of $\mathcal{G}(H^*)$ on $\mathcal{G}(P)$; this fact allows one to obtain the groupoid integrating P/H by symplectic reduction. The result coincides with Theorem 12 in our paper when we take $P = G$ as a Poisson Lie group and $H \subset G$ a Poisson Lie subgroup. It would be nice to extend the results of [21] to the

most general coisotropic reduction described in Theorem 10 in order to get our Theorem 12 in full generality as a particular case of this reduction scheme.

References

- [1] F. Bonechi, A. S. Cattaneo and M. Zabzine: Geometric quantization and non-perturbative Poisson sigma model. *Adv.Theor.Math.Phys.*10:683-712, (2006).
- [2] A. Cattaneo and G. Felder: Poisson sigma model and symplectic groupoid. In *Quantization of singular symplectic quotients*, vol.198 of *Progr.Math.*, 61-93, Birkhauser, Basel (2001).
- [3] A. Connes and G. Landi: Noncommutative Manifolds the Instanton Algebra and Isospectral Deformations. *Commun.Math.Phys.* 221 (2001) 141-159.
- [4] A. Connes, M. Dubois-Violette: Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples. *Commun. Math. Phys.* **230** (2001), 539-579.
- [5] M. Crainic and R. L. Fernandes: Integrability of Lie brackets. *Ann.of Math.* 2 **157(2)** (2003), 575-620.
- [6] V. G. Drinfel'd: On Poisson homogeneous spaces of Poisson-Lie groups. *Theo. Math. Phys.* **95** 2 (1993), 226-227.
- [7] R. L. Fernandes, J. P. Ortega, T. S. Ratiu: The momentum map in Poisson geometry. [arXiv:0705.0562].
- [8] E. Hawkins: A Groupoid Approach to Quantization. [arXiv:math/0612363].
- [9] M. V. Karasev: Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets. *Izv. Akad. Nauk SSSR Ser. Mat.*, **50**, 638, (1986), 508-538. English translation: *Math. USSR-Izv.* **28** 3 (1987), 497-527.
- [10] S. Klimek and A. Leśniewski: Quantum Riemann Surfaces I. The Unit Disc. *Comm. Math. Phys.* **146**, 103-122 (1993).
- [11] L.I. Korogodsky: Quantum Group $SU(1,1) \times \mathbb{Z}_2$ and "Super-Tensor" Products, *Comm.Math.Phys.* **163** (1994), 433-460.
- [12] J.-H. Lu: "*Multiplicative and affine Poisson structures on Lie groups*". Phd thesis, Univ. of California, Berkeley, (1990).

- [13] J.-H. Lu: A note on Poisson homogeneous spaces. [arXiv:0706.1337].
- [14] J.-H Lu and A. Weinstein: Groupoides symplectiques doubles des groupes de Lie-Poisson. C.R.Acad.Sc.Paris, **309** (1989), 951-954.
- [15] J.-H Lu and A. Weinstein: Poisson Lie groups, dressing transformations and Bruhat decompositions. J.Differential Geometry **31** (1990) 501-526.
- [16] K.C.H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*. Cambridge University Press, (2005).
- [17] S. Majid: Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations. Pac.J.of Math. **141** 2, (1990), 311-332.
- [18] K. Mikami and A. Weinstein: Moments and Reduction for Symplectic Groupoids. Publ. RIMS. Kyoto Univ. **24** (1988), 121-140.
- [19] M. Rieffel, Deformation quantization for actions of \mathbb{R}^d , Mem. Amer. Math. Soc. **106** (1993).
- [20] M. A. Semenov-Tian-Shansky: Dressing transformations and Poisson Lie group actions. Publ. RIMS, Kyoto Univ. **21** (1985), 1237-1260.
- [21] L. Stefanini: Integrability and reduction of Poisson group actions. arXiv:0710.57532 [math.SG].
- [22] A. Weinstein: Symplectic groupoids and Poisson manifolds. Bull. Amer. Math. Soc.(N.S.) **16** (1987), 101-104.
- [23] A. Weinstein: Coisotropic calculus on Poisson manifolds. J. Math. Soc. Japan **40** (4) (1988), 705-726.
- [24] A. Weinstein: Symplectic groupoids, geometric quantization and irrational rotation algebras, in *Symplectic geometry, groupoids and integrable systems*. Springer NY (1991), 281-290.
- [25] A. Weinstein, P. Xu: Classical solutions of the quantum Yang-Baxter equations. Commun.Math.Phys. **148** (1992) 309-343.
- [26] P. Xu: Symplectic groupoids of reduced Poisson spaces. C.R.Acad.Sci.Paris Serie I **314** (1992), 457-461.

- [27] P. Xu: Poisson manifolds associated with group actions and classical triangular r -matrices. J. Funct. Anal. **112** (1993), 218-240.
- [28] P. Xu: Dirac submanifolds and Poisson involutions. Ann. Sci. Ecole Norm. Sup. (3) **36** (2003), 403-430.
- [29] P. Xu and A. Weinstein: Extension of symplectic groupoids and quantization. J. Reine Angew. Math. **417** (1991), 159-189.